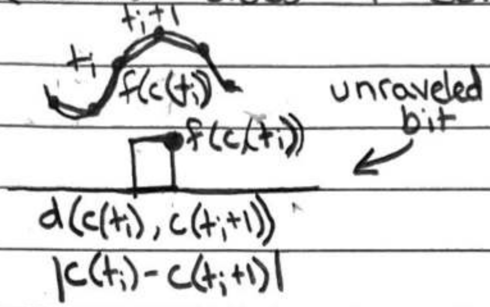
16.2 - 16.3 : Line Integrals

Idea: Given a curve $c : [a,b] \to D \subseteq \mathbb{R}^n$ and a function $f : D \to \mathbb{R}$

$$\mathbb{R} \xrightarrow{c} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$$

How does f behave along the curve?
(What does f contribute?)



1) Piecewise approximation @ c
2) "Unravel" the approximation
3) Above each tiny interval, you get a rectangle/height f (left endpoint)
4) Add the approximations of these rectangles

Definition: The line integral (or path integral) of function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ along curve c parameterized by $\vec{r} : [a,b] \to D$ is

$$\int_C f \, ds = \int_{t=a}^{b} f(\vec{r}(t)) (\vec{r}\,'(t)) \, dt$$

"integral of f along c w.r.t. arc length"

arc length of c

Remark: If $f(\vec{r}) = 1$ for all $\vec{r}$, then $\int_C 1 \, ds = \int_{t=a}^{b} |\vec{r}\,'(t)| \, dt = s(c)$

Ex: Compute $\int_C f\,ds$ for $f(x,y) = 2 + x^2 y$ along C, the upper half of the unit circle with positive orientation. (counterclockwise orientation)

Solution: $\int_C (2 + x^2 y)\,ds$

C parameterized by $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le \pi$

$|\vec{r}\,'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$

$$= \int_{t=0}^{\pi} (2 + \cos^2(t)\sin(t)) \cdot 1\,dt$$

$$= \int_{t=0}^{\pi} 2\,dt + \int_{t=0}^{\pi} \cos^2(t)\sin(t)\,dt$$

$u = \cos(t)$
$du = -\sin(t)$

$$= 2[t]_0^{\pi} - \int_{t=0}^{\pi} u^2\,du = 2[\pi - 0] - \tfrac{1}{3}[u^3]_{t=0}^{\pi}$$

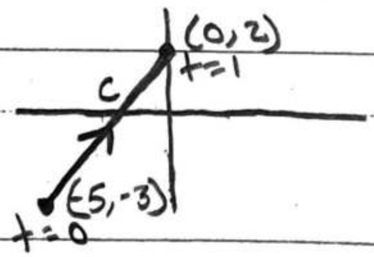
$$= 2\pi - \tfrac{1}{3}[\cos^3(t)]_{t=0}^{\pi} = 2\pi - \tfrac{1}{3}((-1)^3 - (1)^3) = \underline{2(\pi + \tfrac{1}{3})}$$

To measure the "build up" of f-values in one direction x_k, we can use

$$\int_C f\,dx_k = \int_{t=a}^{b} f(\vec{r}(t))\,|x_k'(t)|\,dt$$

where $x_k(t)$ is the x_k-component of $\vec{r}(t)$, a parameterization of C

Ex: Evaluate $\int_C y^2\,dx + \int_C x\,dy$ where C is the line segment oriented from $(-5,-3)$ to $(0,2)$.



Solution: First parameterize C

$\vec{r}(t) = (1-t)\langle -5, -3 \rangle + t\langle 0, 2 \rangle$

Note: Parameterizing a segment from A to B:
$\vec{r}(t) = (1-t)A + tB$

$\vec{r}(t) = (1-t)\langle -5, -3\rangle + t\langle 0, 2\rangle$
$\vec{r}(t) = \langle -5+5t, -3+3t+2t\rangle$
$\vec{r}(t) = \langle -5+5t, -3+5t\rangle$
$\vec{r}'(t) = \langle 5, 5\rangle$
$x(t), y(t)$

$$\therefore \int_c y^2 dx + \int_c x\,dy = \int_{t=0}^{1} (5t-3)^2 \cdot 5\,dt + \int_{t=0}^{1} (5t-5)\cdot 5\,dt$$
$$= \int_{t=0}^{1} 5(5t-3) + 5(5t-5)\,dt = 5\int_{t=0}^{1} (25t^2 - 30t + 9 + 5t - 5)\,dt$$
$$= 5\int_{t=0}^{1} (25t^2 - 25t + 4)\,dt = 5\left[\frac{25}{3}t^3 - \frac{25}{2}t^2 + 4t\right]_{t=0}^{1}$$
$$= 5\left[\left(\frac{25}{3} - \frac{25}{2} + 4\right) - 0\right] = 5\left(-\frac{25}{6} + \frac{24}{6}\right) = 5\left(\frac{-1}{6}\right) = \boxed{-\frac{5}{6}}$$

Definition: The line integral of vector field $\vec{v}$ along curve c parameterized by $\vec{r}(t)$ for $a \le t \le b$ is

$$\int_c \vec{v}(t) \cdot d\vec{r} = \int_{t=a}^{b} \vec{v}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

$\left(= \int_c \vec{v} \cdot \vec{T}\,ds \right.$ where $\vec{T}$ is the unit tangent of $\left.\vec{r}\right)$
$(\vec{T} = \vec{r}'(t)/|\vec{r}'(t)|)$

Ex: Compute $\int_c \vec{v} \cdot dr$ for $\vec{v}(x,y,z) = \langle xy, yz, zx\rangle$ and c the curve parameterized by $\vec{r}(t) = \langle t, t^2, t^3\rangle$ on $0 \le t \le 2$

$\vec{r}'(t) = \langle 1, 2t, 3t^2\rangle$
$\vec{v}(\vec{r}(t)) = \langle t\cdot t^2, t^2\cdot t^3, t^3\cdot t\rangle$
$= \langle t^3, t^5, t^4\rangle$

Solution: $\int_c \vec{v} \cdot dr$

$$= \int_{t=a}^{b} \vec{v}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$
$$= \int_{t=0}^{2} \langle t^3, t^5, t^4\rangle \cdot \langle 1, 2t, 3t^2\rangle$$

$$= \int_{t=0}^{2} (t^3 + 2t^6 + 3t^6)\,dt$$

$$= \int_{t=0}^{2} (t^3 + 5t^6)\,dt = \left[\tfrac{1}{4}t^4 + \tfrac{5}{7}t^7\right]_{t=0}^{2}$$

$$= \left(\tfrac{16}{4} + \tfrac{5 \cdot 128}{7}\right) - 0 = \frac{28+640}{7} = \boxed{\frac{668}{7}}$$

Note: Physics work is just a line integral...
the work done by a particle moving along path $\vec{r}(t)$ for $a \le t \le b$ through vector field $\vec{F}$ is $\int_C \vec{F} \cdot d\vec{r}$

Ex: Compute the work done by particle taking path the clockwise-oriented quarter circle from $(0,1)$ to $(1,0)$ moving through vector field $\vec{F} = \langle x^2, -xy \rangle$

exercise

Think back to the 2nd example:

$$\int_C y^2\,dx + \int_C x\,dy$$

We can abbreviate this type of line integral:

$$\int_C y^2 dx + x dy$$ ← requires integration along the same curve

In general, we abbreviate

$$\int_C P dx + Q dy = \int_C P dx + \int_C Q dy$$

Idea: Line integrals are just one-dimensional integrals which got "twisted up" in n-space
Is there an analogue of the Fundamental Theorem of Calculus for line integrals?

Bad news: Antiderivatives of $f: \mathbb{R}^n \to \mathbb{R}$ don't really make sense... so the answer must be "no" for general "scalar line integrals" $\int_C f\, ds$

Good news: If $\vec{v}$ is a conservative vector field, then its potential functions act like antiderivatives... so there is some hope for conservative vector fields

"FTLI"

Proposition (Fundamental Theorem of Line Integrals):

If C is a smooth curve parameterized by $\vec{r}(t)$ on $[a,b]$ and $f: \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives on C, then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b) - \vec{r}(a))$$

Proof: Using the FTC and the multivariable Chain Rule:

$$\int_C \nabla f \cdot d\vec{r} = \int_{t=a}^{b} \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

by multivariable chain rule $\longrightarrow$ $$= \int_{t=a}^{b} \frac{d}{dt}\left[f(\vec{r}(t))\right] dt$$

by FTC $\longrightarrow$ $$= f(\vec{r}(b) - f(\vec{r}(a))$$

Ex: Compute $\int_C \vec{v} \cdot d\vec{r}$ via the FTLI for $\vec{v} = \langle 3 + 2xy^2, 2x^2y \rangle$ on $\vec{r}(t) = \langle t, \frac{1}{t} \rangle$ for $1 \le t \le 4$

Solution: First compute a potential

$$f(x,y) = \int \frac{\partial f}{\partial x}\, dx$$
$$= \int (3 + 2xy^2)\, dx$$
$$= 3 + x^2y^2 + C(y)$$

$$\therefore 2x^2y = \frac{\partial f}{\partial y}$$
$$= \frac{\partial}{\partial y}\left[3x + x^2y^2 + C(y)\right]$$
$$= 2x^2y + C'(y)$$
$$\therefore C'(y) = 0$$

$\therefore C(y) = D$ for some constant D

$f(x,y) = 3x + x^2y^2 + D$ is a potential for $\vec{v}$ for all D. In particular $D=0$ works and

$\nabla(3x + x^2y^2) = \vec{v}$ $\therefore$ By FTLI we have

$$\int_C \vec{v}\cdot d\vec{r} = f(\vec{r}(4)) - f(\vec{r}(1)) = f(4, \tfrac{1}{4}) - f(1,1)$$

$$= (3\cdot 4 + 4^2(\tfrac{1}{4})^2) - (3\cdot 1 + 1^2(1)^2)$$

$$= \boxed{9}$$